The Minimum Dominating Set Problem on Some Families of Graphs

Tu A. Nguyen* Ngoc C. Lê†

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Abstract

It is well-known that finding minimum dominating set on graphs is a NP-hard problem. In this paper, we tried to answer the question of what cases make the problem become solvable in polynomial time. In particular, we will give concrete proof that a minimum dominating set can be found in polynomial time in two class of graph: $(fork, \overline{P_5})$-free and $(claw, P_5)$-free. In doing so, we also start to develop a new techniques, named reducing set, to tackle minimum dominating set problem in other classes of graphs.

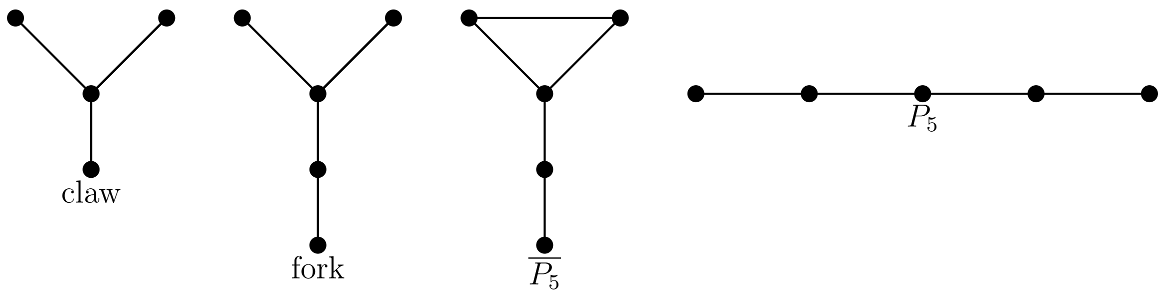
1 Introduction

A dominating set of a graph $G = (V(G), E(G))$ is a set $D \subset V$ of vertices such that every other vertices not in D has at least a neighbor in it. Concretely, the dominating set D of graph G is defined as $D := \{v \in V(G) \mid \forall u \in V(G) \setminus D,\ \exists v \in D \text{ such that } v \in N(u)\}$, where $N(u) := \{v \in V(G) \text{ such that } (u, v) \in E(G)\}$. The minimum dominating set problem asks for such a set with minimum cardinality. This problem is first stated by C.F. De Jaenisch in 1862 when he tried to find the minimum number of queens to dominate a 8x8 chessboard.
The applications of dominating set can be mostly founded in social network theory and communication network [5]. In social network theory, Wang et. al introduced a variation of dominating set, called Positive Influence Dominating Set (PIDS) [7]. A subset D of $V(G)$ is a PIDS if every other node u not in D has at least $deg(u)/2$ neighbor in D. The key idea here is, in modeling social network, nodes in PIDS can be interpreted as positive influencers, and every other individuals should have many positive influcencers (more than half of their friends) so that they will receive mostly positive impact from others. For many reason, such as cost and benefit, we want to find the smallest PIDS. In this research direction, there has been some effort in approximating the minimum PIDS using greedy algorithms [8], [4].
In communication settings, a different variation of dominating set is proposed for a specific problem. For example, Mobile Ad-hoc Network (MANET) asks for the minimum

*Corresponding author. Email: tu.nguyenanh.96@gmail.com
†Email: ngoc.lechi@hust.edu.vn

connected dominating set (CDS) [9] - a dominating set D such that the induced subgraph $G[D]$ is connected. CDS also serves as the backbone for Wireless Sensor Network [2].

In above applications, A good approximation of dominating set is found. However, in this paper, we are interested in finding an exact solution to the minimum dominating set (MDS) problem. In literature, it is well-known that answering whether a graph has a dominating set with cardinality smaller than a given number k is NP-complete. Therefore, we can only hope for algorithm with exponential time complexity. And the best known bound for time complexity is $O(1.4969^n)$, where n is the number of vertices in graph G, due to a measure and conquer approach [6].
Another direction is to answer on which case MDS problem can be solved in polynomial time. We know that independent set and dominating set are closely related: any maximal independent set is minimal dominating set. In literature, there has been extensive research on which family of graph the independent set problem can be solved polynomially by augmenting techniques [3]. A natural idea is to inherit this idea for MDS problem. So instead of finding an augmenting structure, we will look for a "reducing" structure. In this article, we are going to prove that minimum dominating set can be solved in polynomial time in two family of graph $(fork, \overline{P_5})$-free and $(claw, P_5)$-free. Let $G = (V, E)$ is a graph with V is the set of vertices and E is the set of edge. S is a subset of V. We call induced subset S of G, denote $G[S]$ is a subgraph of G such that with vertices set S and has edge (u, v) if and only if $(u, v) \in E(G)$.



If F is set of subset of graph, we called G is F-free if G does not contain any graph in F as proper induced subgraph

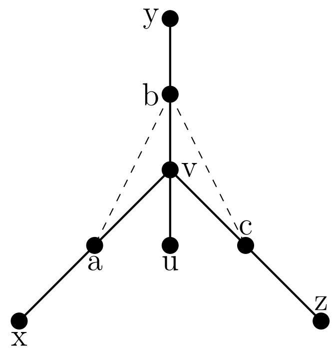
2 $(fork, \overline{P_5})$ - free Graph

In this section, we will point out that beside a special case, minimum dominating set of $(fork, \overline{P_5})$ - free Graph has less than 6 vertices. We will show that by proving the connecting dominating set also has less than 6 vertices. For the sake of concise arguments, if not specify directly, every graph is $(fork, \overline{P_5}) - free$

Lemma 1. *The minimal connected dominating set of a $(fork, \overline{P_5}) - free$ graph can only be path, cycle or clique.*

Proof. Let D be the minimal connected dominating set of G . We will prove that if $v \in V(D)$ satisfies $d_D(v) \geq 3$, then all neighbors of v is connected together.
Suppose $d_D(v) \geq 3$, and let a, b, c be three distinct neighbor of v. If all neighbors of a are adjacent to b, c or v, then we can remove a from D and obtain smaller connected

dominating set (which contradict the assumption about minimality of D). By similar argument, we can conclude that exist $x \in N(a)$, $y \in N(b)$, and $z \in N(c)$ such that, x is not adjacent to b, c, v, y is not adjacent to a, c, v, and z is not adjacent to a, b, v



If a, b, c is not pairwise adjacent, then $\{x, a, v, b, c\}$ induces a fork. Without loss of generality, assume that b and c are adjacent, then $\{x, a, v, b, c\}$ induce $\overline{P_5}$. Suppose a and b are adjacent, then there must exist another vertex $u \in N(v)$ such that u is not adjacent to any of vertices in $\{a, b, c\}$ otherwise remove v out of D create smaller connected dominating set.

If a, c is not adjacent, then $\{x, a, v, u, c\}$ and z, c, v, a, u both induce forks, therefore u must be adjacent to x and z. However, if so, $\{x, u, v, b, c\}$ creates $\overline{P_5}$. Therefore, if $G[D]$ has a vertex of degree 3 or higher then all of its neighbors are connected together. $\square$

With this lemma, we can find minimum connected dominating set by finding minimum dominating clique and finding minimum dominating path or cycle. We can easily check if there is a connected dominating set with cardinality smaller than 3, therefore from now on, we assume that every dominating set has more than 3 elements.

To find minimum dominating clique, we consider each vertex and search for the minimum dominating clique contains that vertex. Let $v \in V(G)$, if $C = v \cup N(v)$ is not a dominating set then we can conclude that v does not belong to any dominating clique and continue our search to another vertex. If C is dominating set of G, then we sequentially remove vertex of C until we obtain a minimal dominating containing v in C. The following lemma show that we can find the minimum dominating set containing v in polynomial time.

Lemma 2. *IF $C = v \cup N(v)$ is a dominating set, $M \subset C$ such that M is minimal dominating set containing v, then M is minimum dominating set contain v*

Proof. Suppose M is not minimum dominating set containing v and contained in C, then there must exist set M' such that $v \in M' \subset M$ and $|M'| < |M|$. For each vertex $x \in M$, let $N_M^r(x) = \{u \in N(x) | u \notin N(x')\ \forall x' \in x,\ x' \neq x\}$ be the neighbor of x such that no other vertex in M is adjacent to. Since we assumed that every dominating set has cardinality greater than 3, and by contradictory supposition, $|M| > 4$. We also have that $N_M^r(x) \neq\ \ \forall x \in M,\ x \neq v$, otherwise we can remove x form M and obtain smaller

dominating set containing v.
Now, we will prove that $|N^r_M(x)| = 1\ \forall x \in M,\ x \neq v$. By contradiction, suppose $\exists x \in M$ such that $N^r_M(x) \geq 2$, let two distinct vertices $x_1, x_2\ \in N^r_M(x)$. Let x' be any vertex belonging to M and different from x and v, and $u \in N^r_M(x')$. We have
x' and x must be adjacent, otherwise x', v, x, x_1, x_2 induces fork or $\overline{P_5}$
u is either adjacent to x_1 or x_2 (or both), otherwise u, x', x, x_1, x_2 creates fork or $\overline{P_5}$
Without loss of generality, suppose $ux_1 \in E(G)$, then u, x_1, x, v, x' induces $\overline{P_5}$, which contradicts our assumption
Since $|M'| < |M|$, then there must exist $x' \in M'$ and $x_1, x_2 \in M$ such that x' is adjacent to both $N^r_M(x_1)$ and $N^r_M(x_2)$. If exists $x_3 \in M, x_3 \neq x_1, x_2$ is not adjacent to x', then $x_3, v, x', N^r_M(x_1), N^r_M(x_1)$ creates a fork. Therefore $\forall x_3 \in M, x_3 \neq x_1, x_2$ then x_3 is adjacent to x'. Moreover, if exists $x_3 \in M, x_3 \neq x_1, x_2$ and x' is not neighbor of $N^r_M(x_3)$ then $N^r_M(x_3), x_3, x', N^r_M(x_1), N^r_M(x_2)$ induces a fork, which means that x_1, x_2, x' is a dominating set. This contradicts our assumption about cardinality of minimum dominating set. □

To this point, we have shown that, minimum dominating clique can be found in polynomial time. To show that MDS can be solved in P-time, by lemma 1, we need to devise a method to find minimum dominating path and cycle. To do that, we consider the following cases. If G is $P_8 - free$ finding minimum dominating path and cycle means search all subgraphs with less than 8 vertices. If G is claw-free, we have the following lemma.

Lemma 3. *If G is claw, $\overline{P_5}$-free and G is not path or cycle then $P_8 - free$*

Proof. The above lemma can be proved by contradiction. Suppose $P = x_1 - x_2 - ... - x_n (n \geq 8)$ be longest path of G. Since G, by assumption, is other than path or cycle, there must exist $y \notin P$ such that y is adjacent to some x_i with $2 < i < n$
Without loss of generality, we can assume that $i \leq 4$ and y is no adjacent to any x_j with $1 < j < i$. We have
x_{i-1}, x_i, x_{i+1}, y creates a claw if y is no adjacent to x_{i+1} $y, x_i, x_{i+1}, x_{i+2}, x_{i+3}$ induces $\overline{P_5}$ if y is not adjacent to x_{i+2} or x_{i+3}. However, in either case, there is a claw with y being center. □

By lemma 3, we are left with dealing the case where G contains both *claw* and P_6. We will iteratively remove vertices in G until at some step k, G^k is either *claw*-free or P_6-free, and dominating set of G^k can be used to construct dominating set of G. To do so, we rely on the following theorem [1]

Theorem 4. *If G is $fork$-free, contains both claw and P_6 as induced subgraph, then there exists a polynomial algorithm that partition $V(G)$ into 3 subsets A, B and C which satisfies all of the following:*

1. *$G(B)$ contains P_6 as induced subgraph*
2. *Any vertices in A is connected to every vertices in B*

3. *There is no edge which has endpoints in B and C*

Lemma 5. *Minimum connected dominating set of $G[A \cup C]$ is also minimum connected dominating set of G*

Let D be the minimum dominating set of $G[A \cup C]$. If $\exists\ v \in\ D$ and $v \in A$, then by Theorem 1, D is also dominating set of D.
Suppose $D \subset C$. If $|D| = 1$, then any vertex of A union with D will create a dominating set of G which contradict the assumption that G has minimum dominating set with cardinality greater than 3. Therefore $|D| \geq 2$
We will prove that every vertices in A dominates C. Let $v \in A$, we prove that v is adjacent to every vertices in C. Since D is dominating set, there must exist $d_1 \in D$ such that d_1 is adjacent to v. Let d_2 be any neighbor of d_1 amd $b_1, b_2 \in B$, v must be adjacent to d_2, since otherwise $\{d_2, d_1, v, b_1, b_2\}$ creates $fork$ or $\overline{P_5}$. By induction and D is dominating set, we conclude that v is adjacent to every vertices in C. However, in this case, every single vertex belonging to A is a dominating set. □

By theorem 1, we can do the following procedure to obtain minimum dominating set. We first partition G into 3 subset A, B and C. Let $G^1 = G[A \cup C]$. G^1 is either *claw* or P_6-free then we can easily find minimum dominating set in of G by lemma 4. Otherwise, we continue partition G^1 into 3 partitions A^1, B^1, C^1, let $G^2 = G^1[A^1 \cup C^1]$, and repeat the above step. By theorem 1, there is at least 6 vertices in B, so each iteration will reduce at least 6 vertices. So after at most $k = [n/6]$ step, G^k will be P_6-free, and by lemma 4, we have that dominating set of G^{k+1} is also dominating set of G^k. Finally, we can state our main theorm of this section.

Theorem 6. *If G is $(fork, \overline{P_5})$-free then the minimum dominating set can be solved in polynomial time.*

3 $(claw, P_5)$ - free Graph

In this section, we will introduce the concept of reducing set to point out that minimum dominating set can be found in polynomial time in $(claw, P_5)$-free class. For brevity, every graph mentioned in this section is $(claw, P_5)$-free.
We can easily prove the following lemma by using the same idea from lemma 1.

Lemma 7. *Every connected component of minimal dominating set is clique*

□

Let D_1 be the minimal dominating set of G. Denote $\{C_1^1, C_2^1, ..., C_{d_1}^1\}$ be connected components of D_1 (here, we suppose that D_1 has more than 1 components i.e $d_1 > 1$). If D_1 is not minimum, there must exist another dominating set D_2 has smaller cardinality $|D_2| < |D_1|$. We also denote $\{C_1^2, C_2^2, ..., C_{d_2}^2\}$ be connected components of D_2.
We will say two components C_i^1 and C_j^2 adjacent if $\exists\ v_1 \in C_i^1$ and C_j^2 such that v_1 coincides v_2 or v_1 is adjacent to v_2. By definition of connected components, it is easily seen that two different components of the same dominating set are not adjacent. Therefore,

we only say adjacent components when one component belongs to a dominating set, and the other one belongs to another dominating set.
Let $R = G[D_1 \cup D_2]$, and denote $R_1, R_2, ..., R_k$ be connected components fo G. Since $|D_2| < |D_1|$, there must exists R_i where the number of vertices belonging to D_1 less that that of which in D_1. We call such components is reducing set, and show that if from D_1 we replace $D_1 \cap R_i$ by $D_2 \cap R_i$, we will obtain smaller dominating set.

Lemma 8. *Let D be a minimal dominating set of G, $a, b \in D$ be two adjacent vertices. Replacing $\{u, v\}$ by $\{u, b\}$ or ($\{v, a\}$) obtains another dominating set D' with the same cardinality as D*

Proof. We only need to prove $D \cup u \setminus a$ a dominating set.
Let $x \in N(u)$ and $x \notin \{a, v\}$. There must be a vertex x in neighbor of u and different form a and v because if $N(u) = \{a, v\}$ then replacing $\{u, v\}$ by $\{v, a\}$, we obtain D' with satified properties.
If $\forall x \in N(u)$, $x \notin \{a, v\}$, x is adjacent to v, then replacing $\{u, v\}$ by $\{v, a\}$ also creates dominating set D' such that $|D'| = |D|$
If $\exists x \in N(u)$, $x \notin \{a, v\}$, x is not adjacent to v, then x must be adjacent to a otherwise $\{u, a, x, v\}$ induces a claw. In this case replacing $\{u, v\}$ by $\{v, a\}$ also creates dominating set D'
Similar argument can be made for $\{u, b\}$ □
We will point out that, every minimal dominating set in this class has a independent dominating set with less or equal cardinality. Inspired by augmenting technique, to deal with finding minimum dominating set, we will start with a minimal dominating set D_1, and keep reducing the number of vertices in D_1 until we are longer able to. Suppose D_1 has a component with more than 1 node, denote $u, v \in D_1$ then by lemma 6, we can replace u, v by u, b or v, a without increasing number of node and keep the dominating properties. This replacement step always creates a connected component with single vertex, since $a \in N^r_D(u)$ (or $b \in N^r_D(v)$). However, above argument can only be true if there is no components in D_1 which contains only a single vertex v and all of its neighbor is adjacent to other vertex in D_1. In this case, we can replace v by one of its neighbor. If the dominating obtained after the replacement step has a non-clique connected component then it is not minimal, we can continue to remove vertices in D_1 to obtain smaller one.
We now assume that every connected components of D_1 has more than 2 vertices. Let C^1_i is a components of D_1 and $C^1_i = \{x^i_1, x^i_2, ..., x^i_{k_i}\}$ ($k_i \leq 2$). In the replacement step, we replace C^1_i by $\{x^i_1, y^i_2, ..., y^i_k\}$ where $y^i_j \in N^r_D(x_j)\ \forall\ j \in \{2, 3, ..., k_i\}$

Lemma 9. *After perform series of replacement steps, we obtain a dominating independent set*

Proof. Suppose after the replacement step of two connected components C^1_i and C^1_j, we obtain dominating set D_2 there exists edge between $\{x^i_1, y^i_2, ..., y^i_{k_i}\}$ and $\{x^j_1, y^j_2, ..., y^j_{k_j}\}$. By the minimality of D_1, we have that x^i_1 and x^j_1 are the two new connected of components D_2. Therefore, there can only be edge between $y^i_{n_i}$ and $y^j_{n_j}$ for some $2 \leq n_i \leq k_i$ and $2 \leq n_j \leq k_j$. However, in this case $x^i_{n_i}, y^i_{n_i}, y^j_{n_j}, x^j_{n_j}, x^j_1$ induce P_5. This means that after

the replacement step, the size of the dominating set will not increase while the number of connected components increase by at least the number of components of D_1.
If after some replacement step, we obtain D_i such that $v \in D_i$ is a component and $N^r_{D_i} = \emptyset$, then by similar argument, we can replace v by one of its neighbor and keep reduce D_i until it is minimal.
If D_i is not independent, we remove all single component and its neighbor from G and D_i, and repeat the replacement step. Since the number of components increase after every replacement step, after finite step, we will obtain an independent dominating set. □

This lemma tells us that, for every minimal dominating set in G there exist an independent set with equal or smaller size. Hence, to find minimum dominating set, we focus on finding minimum independent dominating set. Let $D_1 = \{x^1_1, x^1_2, ..., x^1_{k_1}\}$ and $D_2 = \{x^2_1, x^2_2, ..., x^2_{k_2}\}$ be two minimal independent dominating set and $R = G[D_1 \cup D_2]$. Since D_1 and D_2 are independent, no vertices within the two set is adjacent to each other. Furthermore $\Delta(R) \leq 2$, since if $v \in R$ and $d(v) \leq 3$, suppose $v \in D_1$, then $N_R(v) \subset D_2$, therefore v combined with its neighbors in R create a claw. This means that every connected component of R can only be path or cycle. Moreover, G is P_5-free, every components of R cannot have more than 5 vertices.

Lemma 10. *If C is a connected component of R, denote $C = C_1 \cup C_2$ where C_1 and C_2 are respectively connected components of D_1 and D_2, then $D_1 \setminus C_1 \cup C_2$ is a dominating set.*

Proof. Since C is a connected component with more than 2 vertices, suppose x_1 and x_2 belongs to C and adjacent to each other, where $x_1 \in C_1$ and $x_2 \in C_2$. Without loss of generality, assume contradictory that $\exists x \in G \setminus N[D_1 \setminus C]$ such that x is adjacent to x_1 but not x_2. However, since D_2 is also dominating set, there must exist y_2 does not belong to C and has x as its neighbor. Since D_1 is also a dominating set, and by maximality of C, there must be a vertex $y_1 \in D_1$ but not in C such that y_1 is adjacent to y_2 but not x. In this case x_2, x_1, x, y_2, y_1 induces P_5. □

We are now able to state the main theorem in this section.

Theorem 11. *Minimum doninating set in $(claw, P_5)$-free graph can be found in polynomial time.*

Proof. From Lemma 9, we know that, in $(claw, P_5)$-free graph, there exists an independent dominating has the minimum cardinality. We begin our algorithm by finding a minimal independent dominating set. By lemma 10, we have that a connected component of the union of two minimal dominating set can serve as reducing set. Moreover, sine G is $claw$-free, the bipartite graph can only be path or cycle. P_5-free property make the connected components cannot have more than 5 vertices. Therefore, we can enumerate all path with length three and all cycles with length five. If no path or cycles founded can reduce the number of the current dominating set, we conclude that the minimum dominating set is found. Since after each step, the number of vertices decrease at least one, so after at most n step, an minimum dominating set is found.

4 Conclusion

In this paper, we proved that MDS problem can be solved in polynomial time in two family of graph: $(fork, \overline{P_5})$-free and $(claw, P_5)$-free. In both case, we first want to characterize the property of minimal dominating set and then apply the reducing set technique. However, in the first family, the property of minimal connected dominating set alone help us devise a polynomial algorithm. The second family utilize the concept of reducing set. In fact, the class can be extend to any family of graph that forbid longer path as induced subgraph.

References

[1] V. E. Alekseev. Polynomial algorithm for finding the largest independent sets in graphs without forks. *Discrete Appl. Math.*, 135(1–3):3–16, January 2004.

[2] Ding-Zhu Du and Peng-Jun Wan. *Connected Dominating Set: Theory and Applications*. Springer Publishing Company, Incorporated, 2012.

[3] Alain Hertz and Vadim V Lozin. The maximum independent set problem and augmenting graphs. In *Graph Theory and Combinatorial Optimization*, pages 69–99. Springer, 2005.

[4] Hassan Raei, Nasser Yazdani, and Masoud Asadpour. A new algorithm for positive influence dominating set in social networks. In *Proceedings of the 2012 International Conference on Advances in Social Networks Analysis and Mining (ASONAM 2012)*, ASONAM '12, page 253–257, USA, 2012. IEEE Computer Society.

[5] A Sasireka and AH Nandhu Kishore. Applications of dominating set of graph in computer networks. *International Journal of Engineering Sciences & Research Technology (IJESRT), Vol. No. 3, Issue No*, 1:170–173, 2014.

[6] Johan M. M. Van Rooij and Hans L. Bodlaender. Exact algorithms for dominating set. *Discrete Appl. Math.*, 159(17):2147–2164, October 2011.

[7] Feng Wang, Erika Camacho, and Kuai Xu. Positive influence dominating set in online social networks. In *Proceedings of the 3rd International Conference on Combinatorial Optimization and Applications*, COCOA '09, page 313–321, Berlin, Heidelberg, 2009. Springer-Verlag.

[8] Feng Wang, Hongwei Du, Erika Camacho, Kuai Xu, Wonjun Lee, Yan Shi, and Shan Shan. On positive influence dominating sets in social networks. *Theor. Comput. Sci.*, 412(3):265–269, January 2011.

[9] Jie Wu, Mihaela Cardei, Fei Dai, and Shuhui Yang. Extended dominating set and its applications in ad hoc networks using cooperative communication. *IEEE Trans. Parallel Distrib. Syst.*, 17(8):851–864, August 2006.